Examples of bundles on Calabi-Yau 3-folds for string theory compactifications

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Witten asked in 1997 (and no doubt much earlier too) if there was an example of a rank four bundle E on a Calabi-Yau 3-fold X satisfying the following conditions:

- X is not simply-connected,
- $c_1(E) = 0$; in fact $\Lambda^4 E \cong \mathcal{O}_X$,
- the holomorphic Euler characteristic $\chi(E)$ of E equals 3,
- E is slope-polystable (a direct sum of slope-stable bundles of degree zero with respect to the Kähler form), i.e. admits an $SU(4)$ Hermitian-Yang-Mills connection by the Donaldson-Uhlenbeck-Yau theorem,
- the pairing $H^1(\Lambda^2 E) \otimes H^1(E) \otimes H^1(E) \to \mathbb{C}$ (given by cup-product, wedging, and $H^3(\Lambda^4 E) \cong H^3(\mathcal{O}_X) \cong \mathbb{C}$) should be non-zero,
- the same pairing $H^1(\Lambda^2 E \otimes \alpha) \otimes H^1(E \otimes \beta) \otimes H^1(E \otimes \gamma) \to \mathbb{C}$, where α, β, γ are representations of the fundamental group of X such that $\alpha \otimes \beta \otimes \gamma \cong \mathcal{O}_X$, vanishes for α, β and γ non-trivial,
- the last two conditions also hold with E replaced by E^* throughout,
- one can analyse the pairings $H^1(\text{End } E) \otimes H^1(E) \otimes H^1(E^*) \to H^3(\mathcal{O}_X) \cong \mathbb{C}$, and
- $c_2(X) - c_2(E)$ is $c_2(F)$ for some slope-polystable bundle F of any small rank and $c_1(F) = 0$.

This has something to do with compactifying a 10-dimensional string theory on X to the supersymmetric standard model in four dimensions to study the half-life of the proton. E and F are to be embedded in E_8 bundles, and $\chi(E) = 3$ is the "generation number" – the number of families of quarks. This is the limit of my understanding, but fortunately Witten distilled the physics down to the above purely mathematical question.

Physicists usually concentrate on producing examples satisfying as many of the topological conditions as possible (see [DOPW] for the current state of the art), hoping that if the moduli are big enough there will be at least one bundle satisfying the conditions on the pairings. (The first example satisfying these topological conditions was given by Tian and Yau in [Y] [TY], but their example with $E = TX \oplus \mathcal{O}$ does not satisfy the other constraints.) Physicists also tend to use the Friedman-Morgan-Witten method of constructing bundles on elliptically fibred 3-folds. Here the Serre construction is used; in fact even though the 3-folds below are elliptic the FMW method *does not apply* because the bundles *are not stable on the elliptic fibres* and the nonexistence results of physicists (e.g. [DLOW] rules out these 3-folds) do not apply directly. Here the problem is tackled from the other end, constructing bundles satisfying the pairing conditions (which I feel ought to be the most difficult) with enough freedom to try and control the topological ones. This is only partially successful – the last condition still eludes me (and so [DLOW] may yet apply to this case). $c_2(E)$ is kept as low as possible to make satisfying the last condition feasible, but I have yet to find the required F. The necessary condition (the Bogomolov inequality $c_2(F) \,.\, \Omega \geq 0$, where Ω is the Kähler form) for the existence of a stable F is easily satisfied, but $c_2(F)$ is *not* effective. I still believe the class $c_2(F)$ may be represented by a stable F but my attempts to find one over the last 2 years have failed. (Since it is usually assumed in the physics literature that c_2 of a stable bundle, with $c_1 = 0$, must be an effective curve we give an example to show that this need not be true.)

The first example below is on $K3 \times T^2$; when I showed this to Witten another condition was promptly added to the above list – that X should not be $K3 \times T^2$ (in fact that the holonomy of X should be bigger than $SU(2) \subset SU(3)$). It is included below and worked out in full, however, as it displays most of the ideas of, and is good motivation for, the other two examples, which we run through more briefly as most of the principles are the same. These second two examples take place on the $SU(2) \times \mathbb{Z}_2$ holonomy manifold $(K3 \times T^2)/\mathbb{Z}_2$ where the $\mathbb{Z}_2$-action is an Enriques action on the $K3$ times by -1 on T^2, and so is free and preserves the canonical class. The bundles we find are not really full $SU(4)$-bundles, but have smaller structure group $U(2) \times_{U(1)} U(2) \subset SU(4)$ (i.e. they are direct sums of rank two bundles of opposite c_1); I think Witten's intention was to deform them to non-split bundles – the pairings involving the deformation space $H^1(\operatorname{End} E)$ would enable one to study such deformations. This paper is inevitably a rather dry list of mathematical constructions, but the general technique used to satisfy the conditions on the pairings should be clear from the first example.

Acknowledgements. Many thanks to Edward Witten for discussing the problem with me several times and allowing me to use his results and intuition; I have probably misquoted him and of course have not solved the problem anyway. Thanks

to Ian Graham for the use of his house in the south of France, and to Ian Dowker and Ivan Smith for $K3$ advice. This work was mainly done at the Institute for Advanced Study in 1997–98 under NSF grant DMS 9304580. Having finally given up on solving the last condition, a broken man, I have decided to circulate this in the hope that someone else might use the same techniques to better effect. I am grateful to Professors Yau and Taubes for their current support at Harvard University where this paper was written up. Thanks to Tony Pantev for his encouragement and interest.

1 $K3 \times T^2$

Let S be a smooth $K3$ $(2,3)$-divisor in $\mathbb{P}^1 \times \mathbb{P}^2$. Let ω_1, ω_2 be (the restrictions to S of) the pullbacks to $\mathbb{P}^1 \times \mathbb{P}^2$ of the Fubini-Study Kähler forms on $\mathbb{P}^1$, $\mathbb{P}^2$ respectively. They are the first Chern classes of the line bundles $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$, in the obvious notation. Tensor powers of these give the line bundles $\mathcal{O}(i,j)$. Note that $\omega_1^2 = 0,\ \omega_1\omega_2 = 3,\ \omega_2^2 = 2$, and that ω_2 is a nondegenerate Kähler form on S.

The line bundle $L = \mathcal{O}(-1,1)$ is of degree -1 (with respect to ω_2), has $c_1(L)^2 = -4$, and it and its dual are acyclic – that is

$$H^i(L) = 0 = H^i(L^*) \quad \forall i.$$

This can be seen by simple exact sequences on $\mathbb{P}^1 \times \mathbb{P}^2$, or from the obvious fact that L and L^* have no sections, and Riemann-Roch.

We are going to define two rank 2 bundles on S, with determinant L and L^* respectively, by the Serre construction on a surface ([GH] p 726, [DK] Chapter 10), which we briefly describe now. Just as (codimension 1) divisors correspond to (rank 1) line bundles, codimension 2 (i.e. dimension zero, on S) subschemes Z sometimes correspond to rank 2 bundles E via zero sets of sections $s \in H^0(E)$. Suppose we have such an s with zero locus Z. Then, just as wedging with a non-zero vector $v \in V$ in a 2-dimensional vector space V gives an exact sequence $0 \to \mathbb{C} \to V \to \Lambda^2 V \to 0$, wedging with $s \in H^0(E)$ gives a sequence of sheaves $0 \to \mathcal{O}_X \to E \to \Lambda^2 E \to 0$ which is exact away from the zeros of s. If s vanishes only in codimension 2 (i.e. not on a divisor) this sequence is in fact globally exact except at the last term, where it is clearly onto only those sections of $\Lambda^2 E$ that vanish on Z, so giving the exact sequence

$$0 \to \mathcal{O}_X \to E \to L \otimes \mathcal{I}_Z \to 0,$$

where $\mathcal{I}_Z$ is the ideal sheaf of functions vanishing on Z, and L is the line bundle $\Lambda^2 E$.

The Serre construction provides a partial converse to this construction: given Z and L one tries to reconstruct an E with $\Lambda^2 E \cong L$ and $s \in H^0(E)$ (with zeros on

Z) as an extension of $L \otimes \mathcal{I}_Z$ by $\mathcal{O}_X$ as in the above sequence. Such an extension is of course given by an element of $H^1(L^*)$ away from Z, which we may think of as an L^*-valued one-form that is $\bar\partial$-closed away from Z. Local analysis on Z ([DK] Chapter 10) shows that the extension E being locally free is equivalent to $\bar\partial$ of the form being some non-zero multiple $a_z\delta_z$ of the Dirac delta at each point z of Z. Thus the condition for the global existence of such a vector bundle E is that the class in $H^2(L^*)$ defined by some combination $\sum_{z\in Z} a_z\delta_z$ $(a_z \neq 0\ \forall z \in Z)$ is $\bar\partial$ of something, i.e. that it is zero in cohomology. (Here the residue data a_z is really an element of the line $L^* \otimes K_X^*|_z$, thus giving a linear functional on $H^0(L \otimes K_X)$ by restriction to z. By Serre duality $H^0(L \otimes K_X)^* \cong H^2(L^*)$ this gives an element of $H^2(L^*)$ as claimed. The a_zs give the dual of the determinant of the derivative $ds \in (T^*X \otimes E)|_z$ of the section $s \in H^0(E)$ at the zeros z.)

One upshot of all this which will suffice for our needs is that *if* $H^2(L^*) = 0$ *then* E *and* s *exist.*

So define the dual A^* by the exact sequence of sheaves

$$0 \to \mathcal{O} \xrightarrow{s_{A^*}} A^* \to \mathcal{I}_2(1,-1) \to 0, \tag{1.1}$$

where $\mathcal{I}_2(1,-1)$ denotes the ideal sheaf of functions vanishing at two fixed points on S, twisted by $L^* = \mathcal{O}(1,-1)$. There is no obstruction to defining bundles in this way (with a section s_{A^*} vanishing exactly at the two points) as $H^2(L) = 0$.

From the above sequence it is evident that

$$\Lambda^2(A) = L, \quad c_2(A^*) = 2 = c_2(A), \quad H^0(A) = 0, \quad H^0(A^*) = \mathbb{C}\,.\,s_{A^*}.$$

Similarly we define B by

$$0 \to \mathcal{O} \xrightarrow{s_B} B \to \mathcal{I}_3(1,-1) \to 0, \tag{1.2}$$

with

$$\Lambda^2(B) = L^*, \quad c_2(B) = 3, \quad H^0(B) = \mathbb{C}\,.\,s_B, \quad H^0(B^*) = 0.$$

Now let T denote an elliptic curve, and set $X = S \times T \xrightarrow{\pi} S$. Finally denote by $\mathcal{O}_T(n)$ the nth power of the pull-back to X of a fixed degree one line bundle on T, with first Chern class $n\omega_T$. Then we can define

$$A' := \pi^*A \otimes \mathcal{O}_T(3), \quad B' := \pi^*B \otimes \mathcal{O}_T(-3), \quad E := A' \oplus B'. \tag{1.3}$$

Assume that S is a generic divisor in $\mathbb{P}^1 \times \mathbb{P}^2$, so that its only line bundles are the $\mathcal{O}(i,j)$s by Noether-Lefschetz theory. Then we have

Theorem 1.4 *The rank 4 bundle E defined in (1.3) satisfies all but the last of the conditions listed at the start of this paper, with respect to the Kähler form $\Omega = \pi^*\omega_2 + 6\omega_T$.*

Proof Throughout we will often suppress pull-backs for clarity; thus A will often denote π^*A and ω_2 will be confused with $\pi^*\omega_2$.

Firstly $\Lambda^4 E \cong \Lambda^2 A \otimes \mathcal{O}_T(6) \otimes \Lambda^2 B \otimes \mathcal{O}_T(-6) \cong L \otimes L^* \cong \mathcal{O}_X$ is trivial. By Riemann-Roch the holomorphic Euler characteristics $\chi(A)$, $\chi(B)$ are 0 and -1 respectively, so that

$$\chi(E) = \chi(\mathcal{O}_T(3))\chi(A) + \chi(\mathcal{O}_T(-3))\chi(B) = 3(\chi(A) - \chi(B)) = 3,$$

as required. The choice of Kähler form $\Omega = \omega_2 + 6\omega_T$ ensures that A' and B' have degree zero, so $E = A' \oplus B'$ is polystable if and only if A' and B' are stable, which in turn is equivalent to A and B being stable on the $K3$ surface S. This follows from

Lemma 1.5 *Recall that S was chosen such that its only line bundles are the $\mathcal{O}(i,j)$s. Then letting P denote either of A^* or B on S, P is stable with respect to ω_2.*

Proof We must show that $P(i,j)$ has no sections for $0 \geq \deg P(i,j) = 6i + 4j + 1$, i.e. for $3i + 2j \leq -1$. But the presentations (1.1, 1.2) give the sequence

$$0 \to \mathcal{O}(i,j) \to P(i,j) \to \mathcal{I}_n(i+1, j-1) \to 0,$$

where n is either 2 or 3 points on S. The degree of $\mathcal{O}(i,j)$ is $3i + 2j \leq -1$, so it has no sections. Similarly since the degree of $\mathcal{O}(i+1, j-1)$ is $3i + 2j + 1 \leq 0$, this line bundle can only have a section if it is trivial and the section has no zeros. Thus $\mathcal{I}_n(i+1, j-1)$ has no sections for $n > 0$, and we have shown that $P(i,j)$ has no sections and so is stable. □

We now turn to the pairings $H^1(\Lambda^2 E \otimes \alpha) \otimes H^1(E \otimes \beta) \otimes H^1(E \otimes \gamma) \to \mathbb{C}$, for α, β, γ flat line bundles with $\alpha \otimes \beta \otimes \gamma \cong \mathcal{O}_X$. This is what motivated the construction of E; the basic idea being that the cohomology of flat line bundles on the elliptic curve T behaves in a way that is reminiscent of Witten's condition on the pairings; namely it is non-trivial if and only if the line bundle is trivial.

Lemma 1.6 *Given representations α, β, γ of the fundamental group of X, such that $\alpha \otimes \beta \otimes \gamma \cong \mathcal{O}_X$, consider the pairing $H^1(\Lambda^2 E \otimes \alpha) \otimes H^1(E \otimes \beta) \otimes H^1(E \otimes \gamma) \to \mathbb{C}$. Then this is non-zero for the representations trivial, and vanishes for α, β and γ non-trivial.*

The same is also true with E replaced by E^* throughout.

Proof $H^1(\Lambda^2 E \otimes \alpha) = H^1(\Lambda^2 A' \otimes \alpha) \oplus H^1(A' \otimes B' \otimes \alpha) \oplus H^1(\Lambda^2 B' \otimes \alpha)$, and the first and last terms vanish by the Künneth formula, since $L = \Lambda^2 A$ and $L^* = \Lambda^2 B$

have no cohomology on S and α, β, γ are pulled up from T. Thus $H^1(\Lambda^2 E \otimes \alpha) = H^1_S(A \otimes B) \otimes H^0_T(\alpha) \ \oplus \ H^0_S(A \otimes B) \otimes H^1_T(\alpha)$ and the pairing reduces to

$$\big[H^1_S(A \otimes B) \otimes H^0_T(\alpha)\big] \ \otimes \ \big[H^1_S(A) \otimes H^0_T(\mathcal{O}_T(3) \otimes \beta)\big] \\ \otimes \ \big[H^0_S(B) \otimes H^1_T(\mathcal{O}_T(-3) \otimes \gamma)\big] \ \to \ \mathbb{C}\,,$$

plus the same with β and γ exchanged. (The pairing involving $\big[H^0_S(A \otimes B) \otimes H^1_T(\alpha)\big]$ vanishes because no $H^1_S(B)$ term survives the Leray spectral sequence to cup it with.)

So we see that for α non-trivial the whole pairing vanishes. (Note that for α trivial but $\beta \cong \gamma^{-1}$ non-trivial the pairing does not vanish; I am not sure if this is relevant for the physics.) For $\alpha = \beta = \gamma = \mathcal{O}$ we are left with showing, then, that the pairing *on* S

$$H^1_S(A \otimes B) \otimes H^1_S(A) \otimes H^0_S(B) \to \mathbb{C}$$

is non-zero. (The full pairing on E is two copies of the tensor product of this with the non-vanishing cup-product $H^0_T(\mathcal{O}_T(3)) \otimes H^1_T(\mathcal{O}_T(-3)) \to \mathbb{C}\,$.)

But this pairing is Serre-dual to $H^1(A) \otimes H^0(B) \to H^1(A \otimes B)$, and $H^0(B) = \mathbb{C}\,.\,s_B$, so it is sufficient to show that the map

$$H^1(A) \xrightarrow{s_B} H^1(A \otimes B)$$

is non-zero.

Tensoring (1.2) with A (recalling that $A \otimes \Lambda^2 A^* \cong A^*$) and taking cohomology gives

$$H^0(A^* \otimes \mathcal{I}_3) \to H^1(A) \xrightarrow{s_B} H^1(A \otimes B).$$

(1.1) shows that $H^0(A^* \otimes \mathcal{I}_3) = 0$ and $H^1(A) \cong H^1(\mathcal{I}_2) \cong \operatorname{coker}[H^0(\mathcal{O}_S) \to H^0(\mathcal{O}_2)] \cong \mathbb{C}\,$, so the second map in the above sequence does not vanish, as required.

The dual pairing, with E replaced by E^*, is similar. As above, since L and L^* have no cohomology on S, and α has no cohomology on T unless it is trivial, the pairing vanishes for α non-trivial, and in the $\alpha = \beta = \gamma = \mathcal{O}$ case it quickly reduces to

$$H^1_S(A^* \otimes B^*) \otimes H^0_S(A^*) \otimes H^1_S(B^*) \to \mathbb{C}\,, \tag{1.7}$$

with $H^0_S(A^*)$ generated by s_{A^*}. Now (1.1) twisted by B^* yields

$$H^0(B \otimes \mathcal{I}_2) \to H^1(B^*) \xrightarrow{s_{A^*}} H^1(A^* \otimes B^*).$$

Using (1.2) we can see that the first group is either $\mathbb{C}$ or 0 (depending on whether or not the 2 points used to define A^* are a subset of the 3 points used to define B) and the second group is $\mathbb{C}^2$. Thus the second map, which is Serre-dual to (1.7), has

rank 1 or 2, and the dual pairing is non-zero also. $\square$

Finally then we want to understand $H^1(\text{End } E) \otimes H^1(E) \otimes H^1(E^*) \to \mathbb{C}$. Using the fact that $H^0_T(\mathcal{O}_T(6)) \cong \mathbb{C}^6$ etc., we can express $H^1(\text{End}\, E)$ in terms of cohomology groups on S as

$$H^1(\text{End}\, A) \;\oplus\; H^1(\text{End}\, B) \;\oplus\; H^0(A^* \otimes B) \otimes \mathbb{C}^6 \;\oplus\; H^1(B^* \otimes A) \otimes \mathbb{C}^6. \qquad (1.8)$$

This is easily computed to be $6+10+1.6+12.6 = 94$-dimensional. But we will find that only the first two groups (16 dimensions) contribute to the pairing. The part of the pairing involving the last group in the above decomposition is

$$H^1((B')^* \otimes A') \otimes H^1(B') \otimes H^1((A')^*) \to \mathbb{C},$$

which, by the Künneth formula, reduces to a pairing on S tensored with the cup-product

$$H^0_T(\mathcal{O}_T(6)) \otimes H^1_T(\mathcal{O}_T(-3)) \otimes H^1_T(\mathcal{O}_T(-3))$$

on T, and this vanishes.

The third group in (1.8) is involved in the cup-product pairing

$$H^0(A^* \otimes B) \otimes H^1(A) \otimes H^1(B^*) \to \mathbb{C}$$

on S (tensored with a pairing on T^2). We will now show that this vanishes.

Tensoring (1.2) with A^* shows that $H^0(A^* \otimes B)$ is spanned by $s_{A^*} \otimes s_B$, so it is enough, by Serre-duality, to show the vanishing of

$$H^1(B^*) \xrightarrow{s_{A^*} \otimes s_B} H^1(A^*).$$

(1.2) twisted by L shows that wedging with s_B gives an isomorphism $H^1(B^*) \xrightarrow{\wedge s_B} H^1(\mathcal{I}_3)$, while (1.1) shows that the quotient of $\mathcal{I}_3 \xrightarrow{s_{A^*}} A^*$ is $\mathcal{I}_2(1,-1) \oplus \mathcal{O}_3$, giving an exact sequence

$$0 \to \mathcal{I}_3 \xrightarrow{s_{A^*}} A^* \to \mathcal{I}_2(1,-1) \oplus \mathcal{O}_3 \to 0,$$

whose cohomology gives the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \arrow[r] & \mathbb{C}^3 \arrow[r] & H^1(\mathcal{I}_3) \arrow[r, "s_{A^*}"] & H^1(A^*) \arrow[r] & \mathbb{C}^2 \arrow[r] & \mathbb{C} \arrow[r] & 0. \\
 & & & H^1(B^*) \arrow[u, "\wedge s_B", "\wr"'] \arrow[ur, "s_{A^*} \otimes s_B"'] & & & &
\end{tikzcd}$$

It is easy to check that the diagram commutes, and $H^0(\mathcal{I}_3) \cong \mathbb{C}^3$, so the map labelled s_{A^*} must be zero; therefore that marked $s_{A^*} \otimes s_B$ also vanishes, as claimed.

So from the decomposition (1.8) we see that the coupling $H^1(\mathrm{End}\ E)\otimes H^1(E)\otimes H^1(E^*)\to\mathbb{C}$ on X reduces to the direct sum of the corresponding couplings for A' and B'. By the Künneth formula and Serre duality we are left with understanding the couplings

$$\begin{array}{ccc} H^1(A)\otimes H^0(A^*)\to H^1(\mathrm{End}\,A) & \text{and} & H^0(B)\otimes H^1(B^*)\to H^1(\mathrm{End}\,B) \\ \mathbb{C}\otimes\mathbb{C}\ \to\ \mathbb{C}^6 & & \mathbb{C}\otimes\mathbb{C}^2\ \to\ \mathbb{C}^{10}. \end{array}$$

Twisting (1.1) by A and taking cohomology gives an exact sequence

$$\ldots\to H^1(A)\xrightarrow{s_{A^*}} H^1(\mathrm{End}\,A)\to H^1(A^*\otimes\mathcal{I}_2)\to\ldots \tag{1.9}$$

Here the first map is injective because the previous two terms in the sequence are $H^0(\mathrm{End}\,A)\to H^0(A^*\otimes\mathcal{I}_2)$, which is $\mathbb{C}\,.\,\mathrm{id}\xrightarrow{s_{A^*}}\mathbb{C}\,.\,s_{A^*}$ and so an isomorphism. (A is stable so has only scalar endomorphisms; similarly the last map can be seen to be onto and the last group isomorphic to $\mathbb{C}^5$.) But this first map is the A-pairing, which is therefore an injection $\mathbb{C}\otimes\mathbb{C}\hookrightarrow\mathbb{C}^6$.

As for B, twist (1.2) by B^* to get, by similar arguments,

$$0\to H^1(B^*)\xrightarrow{s_B} H^1(\mathrm{End}\,B)\to H^1(B\otimes\mathcal{I}_3)\to 0,$$

so that again the pairing is an injection $\mathbb{C}\otimes\mathbb{C}^2\hookrightarrow\mathbb{C}^{10}$. □

So all that is left is to find an F with trivial determinant and $c_2(F)=c_2(X)-c_2(E)=15[T]+6\omega_T(\omega_1-\omega_2)$ (where $[T]$ denotes the class in $H^4(X;\mathbb{Z})$ Poincaré-dual to any torus fibre T). Suppose we try for a rank 2 F, then one can calculate that $c_2(F':=F\otimes L^*\otimes\mathcal{O}_T(3))=11[T]$, which is effective. Similarly if F has rank 3 then $c_2(F':=F\otimes L^*\otimes\mathcal{O}_T(1))=4[T]$. One could therefore try to use these facts to create a stable F, for instance by using the Serre construction to manufacture a G with $\Lambda^2G=L^*$ and $c_2(G)=11[T]$ then twisting by $\mathcal{O}_T(n)$ s and modifying in codimension 1 with elementary transformations, etc, to get the desired F'. All my attempts have produced *unstable* bundles, however.

We note here that it is *not* necessary for $c_2(F)$ to be effective (i.e. represented by a holomorphic curve) for F polystable with $c_1(F)=0$. Indeed the polystable E constructed above (1.3) has $c_1(E)=0$ and $c_2(E)=9[T]-6\omega_T(\omega_1-\omega_2)$. Thus $c_2(E)\,.\,\omega_2=-6$ is negative and $c_2(E)$ cannot be effective.

2 $(K3\times T^2)/\mathbb{Z}_2$

Now let X be the $SU(2)\times\mathbb{Z}_2$ holonomy manifold $(K3\times T)/\mathbb{Z}_2$, where the $K3$ is a universal cover of an Enriques surface $S=K3/\sigma$ and T is an elliptic curve (which

therefore has a zero and a multiplication by -1). Then the $\mathbb{Z}_2$-action is generated by $\sigma \times (-1)$, and so is free (since σ is) and preserves the canonical class (since both σ and -1 act as -1 on it). $X \xrightarrow{\pi} S$ is a T-fibration over S with no singular fibres, monodromy -1 around $\pi_1(S)$, and a section $S \hookrightarrow X$ at $0 \in T$. It is also a $K3$-fibration over $\mathbb{P}^1 = T/\pm 1$ with four singular fibres which are double fibres modeled on S.

Then $\pi_1(X)$ is easily seen to be given as an extension $1 \to \mathbb{Z}^2 \to \pi_1(X) \to \mathbb{Z}_2 \to 1$ (where the $\mathbb{Z}^2$ is $\pi_1(K3 \times T)$). Fixing generators α, β of $\pi_1(T) \hookrightarrow \pi_1(X)$ ($T \hookrightarrow X$ as the fibre of $X \to \mathbb{P}^1$), and letting $\gamma \in \pi_1(S) \hookrightarrow \pi_1(X)$ be the generator of $\mathbb{Z}_2$ ($S \hookrightarrow X$ as the section of $X \xrightarrow{\pi} S$), we have a presentation

$$\pi_1(X) = \langle \alpha, \beta, \gamma \rangle / (\alpha\beta = \beta\alpha,\ \gamma^{-1}\alpha\gamma = \alpha^{-1},\ \gamma^{-1}\beta\gamma = \beta^{-1},\ \gamma^2 = 1)$$

whose abelianisation is $H_1(X;\mathbb{Z}) = \mathbb{Z}_2^3$ (generated by α, β, γ). The corresponding flat line bundles, given by the representation $\{\text{generator} \mapsto (-1) \in U(1)\}$ will also be denoted by α, β, γ. Note that $\gamma = \pi^* K_S$ is the pull-back of the canonical bundle of S to X.

As before pick an acyclic line bundle L of degree -1 on S:

$$H^i(L) = 0 = H^i(L^*) = H^i(L \otimes \gamma) \quad \forall i, \qquad c_1(L)^2 = -2,\ c_1(L) \,.\, \omega = -1,$$

with respect to an *integral* Kähler form ω on S. (For instance on the Enriques surface studied in ([BPV] V.23) with corresponding $K3$ the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over a certain $(4,4)$-curve, the line bundle $\mathcal{O}(-1,1)$ restricted to $K3$ descends to such a line bundle on S. The two Fubini-Study forms on the $\mathbb{P}^1$s pull-back, restrict and descend to integral forms ω_1, ω_2 on S; we then choose $\omega = \omega_1 + 2\omega_2$.)

Next define bundles A and B on S by the Serre construction as before,

$$0 \to \mathcal{O} \xrightarrow{s_{A^*}} A^* \to \mathcal{I}_1 \otimes L^* \to 0, \tag{2.1}$$

and

$$0 \to \mathcal{O} \xrightarrow{s_B} B \to \mathcal{I}_2 \otimes L^* \to 0, \tag{2.2}$$

with $\mathcal{I}_1$ and $\mathcal{I}_2$ the ideal sheaves of a point and a zero-dimensional subscheme of length 2, respectively, where *we take the point to lie in the length 2 subscheme* this time.

We are ready to define

$$A' := \pi^* A \otimes \mathcal{O}(3S) \otimes \gamma, \quad B' := \pi^* B \otimes \mathcal{O}(-3S) \otimes \gamma, \quad E := A' \oplus B', \tag{2.3}$$

where $S \hookrightarrow X$ is the zero-section of $X \xrightarrow{\pi} S$, defining a divisor with corresponding line bundle $\mathcal{O}(S)$.

Theorem 2.4 *The rank 4 bundle E defined in (2.3) satisfies all but the last of the conditions mentioned at the start of this paper, with respect to any Kähler form Ω in the class of $12[S]+\pi^*\omega$.*

Proof By construction $\Lambda^4 E \cong \mathcal{O}_X$, and by Riemann-Roch, or by lifting to $K3\times T$ and dividing by 2,

$$\chi(E)=\chi(A')+\chi(B')=3\chi_S(A)+(-3)\chi_S(B)=3(c_2(B)-c_2(A))=3(2-1)=3,$$

the second condition.

Since by choice of Ω both A' and B' have degree zero, to prove slope-polystability it is enough to check that A and B are slope-stable on S with respect to ω (which was chosen to be integral and such that $c_1(L)\,.\,\omega=-1$, remember).

Letting P be one of A^* or B, of degree 1, to check stability we need only show that $P\otimes\eta$ has no sections for any line bundle η of degree less than or equal to -1 (this is where the integrality of ω is used). But we have a sequence (2.1, 2.2)

$$0\to\eta\to P\otimes\eta\to L^*\otimes\eta\otimes\mathcal{I}\to 0,$$

for $\mathcal{I}$ some non-trivial ideal sheaf. η has no sections since it has degree ≤ -1, and $L^*\otimes\eta$ has degree ≤ 0 so has no sections with zeros. Since $\mathcal{I}$ is non-trivial this shows that P has no sections, as required.

We now turn to the $\Lambda^2 E$ pairing, which (in the untwisted case) splits as

$$\left.\begin{array}{c} H^1(\Lambda^2 A')\otimes H^1(B')\otimes H^1(B') \\ \oplus \\ (H^1(A'\otimes B')\otimes H^1(A')\otimes H^1(B'))^{\oplus 2} \\ \oplus \\ H^1(\Lambda^2 B')\otimes H^1(A')\otimes H^1(A') \end{array}\right\} \longrightarrow \mathbb{C}\,,$$

and similarly for the dual pairing. We compute these using the Leray spectral sequence for $X\xrightarrow{\pi}S$, noting that $\pi_*\mathcal{O}(nS)\cong\mathcal{O}_S^{\oplus n}$ $(n\geq 0)$ and so, by relative Serre duality, $R^1\pi_*\mathcal{O}(-nS)\cong\gamma^{\oplus n}$. (Again we are suppressing some pull-backs and identifying γ with K_S.)

Thus $H^i(\Lambda^2 A')\cong H^i_S(L^{\oplus 6})=0$ and $H^i(\Lambda^2 B')\cong H^{i-1}_S((L^*\otimes\gamma)^{\oplus 6})=0$, and the same holds on twisting by flat line bundles or taking duals. Therefore only the central pairing above survives.

Tensoring (2.2) by A or $A\otimes\gamma$, and using the fact that $H^0(A^*\otimes\mathcal{I}_2)=0$ (since A^* has only one section s_{A^*}, and this vanishes at one point only), we see that

$$H^0(A\otimes B)=0=H^0(A\otimes B\otimes\gamma),$$

and the same also holds with A, B replaced by A^*, B^*. Thus the $H^0_S(R^1\pi_*\mathcal{O}_X \otimes\ .)$ terms that appear in the pairing (from the Leray spectral sequence) vanish, and we are left with two copies of

$$H^1_S(\pi_*\mathcal{O}_X \otimes A \otimes B) \otimes H^1_S((A \otimes \gamma)^{\oplus 3}) \otimes H^0_S(B^{\oplus 3}) \to \mathbb{C}\,,$$

and the dual pairing is twice

$$H^1_S(\pi_*\mathcal{O}_X \otimes A^* \otimes B^*) \otimes H^0_S((A^*)^{\oplus 3}) \otimes H^1_S((B^* \otimes \gamma)^{\oplus 3}) \to \mathbb{C}\,.$$

Twisting by any flat line bundle that is non-trivial on the T fibres destroys the $\pi_*\mathcal{O}_X$ term (this was the original idea for the whole construction of course), so we need only consider twisting the first terms by γ. Since the tensor product of the three line bundles we tensor by must be trivial, one of the other two terms must be twisted by something containing a γ factor (i.e. not in the span of α, β).

Thus the pairings become, by Serre duality, 2.3.3=18 copies of

$$H^1_S(A \otimes \gamma) \otimes H^0_S(B) \to H^1_S(A \otimes B \otimes \gamma), \tag{2.5}$$

and the dual

$$H^0_S(A^*) \otimes H^1_S(B^* \otimes \gamma) \to H^1_S(A^* \otimes B^* \otimes \gamma). \tag{2.6}$$

Twisting either of the H^0 groups above by γ destroys them by inspection of (2.1, 2.2). So from what we have already proved it is now enough to show that the pairings are non-trivial as they stand but trivial when the H^1 groups above are twisted by γ.

For the first case (2.5) note that the required vanishing follows from the vanishing of $H^1_S(A)$: $c_2(A)$ was chosen to be 1 making $\chi(A) = 0 = h^0(A) - h^1(A) + h^0(A^* \otimes \gamma) = -h^1(A)$. The non-triviality of the untwisted pairing follows by taking the cohomology of $A \otimes \gamma \otimes$ (2.2):

$$0 \to H^1(A \otimes \gamma) \xrightarrow{s_B} H^1(A \otimes B \otimes \gamma),$$

where the first zero follows from $H^0(A^* \otimes \gamma) = 0$ (2.1). Since the first group has dimension 1 (also by Riemann-Roch, since $h^2(A \otimes \gamma) = h^0(A^*) = 1$) the pairing (2.5) is non-zero.

For the dual pairing (2.6) we take the cohomology of (2.1) $\otimes B^* \otimes \gamma$, giving

$$0 \to H^0(B \otimes \mathcal{I}_1 \otimes \gamma) \to H^1(B^* \otimes \gamma) \xrightarrow{s_{A^*}} H^1(A^* \otimes B^* \otimes \gamma).$$

The pairing is the second map so we want the first map to *not* be onto, but to be onto when the γ s are removed (so that the twisted pairing vanishes). Recalling that we chose the zeros of s_{A^*} to lie in those of s_B we see that $H^0(B \otimes \mathcal{I}_1) = \mathbb{C}\ .\, s_B$, while

$H^0(B \otimes \mathcal{I}_1 \otimes \gamma) = 0$. Since, by Riemann-Roch, $h^1(B^* \otimes \gamma) = 2$ and $h^1(B^*) = 1$, this gives the required result. $\square$

Again finding a stable F with $c_2(F) = c_2(X) - c_2(E)$ and $c_1(F) = 0$ has defeated me. For F rank 2 this works out as $c_2(F(3S) \otimes L^*) = 5[T]$, for what it's worth.

3 $(K3 \times T^2)/\mathbb{Z}_2$ again

We give a final example, again satisfying all but the last of Witten's conditions. Since the example will be very similar to the last one in all but the pairings (as L^* will no longer be acyclic) we will concentrate mostly on them.

Pick an Enriques surface with a -2-sphere C in it, let $K3$ be its universal cover, and again set $X = (K3 \times T^2)/\mathbb{Z}_2$. We will use the notation of the last section.

Letting $L = \mathcal{O}(-C)$, L and $L^* \otimes \gamma$ are acyclic while

$$H^i(L^*) = \begin{cases} \mathbb{C} & i = 0 \\ \mathbb{C} & i = 1 \\ 0 & i = 2 \end{cases} \qquad \text{and} \qquad H^i(L \otimes \gamma) = \begin{cases} 0 & i = 0 \\ \mathbb{C} & i = 1 \\ \mathbb{C} & i = 2. \end{cases}$$

We can then construct A and B on S by the Serre construction, using the fact that $H^2(L) = 0$ so there is no obstruction to finding locally free sheaves with presentations

$$0 \to \mathcal{O} \xrightarrow{s_{A^*}} A^* \to L^* \otimes \mathcal{I}_1 \to 0, \tag{3.1}$$

and

$$0 \to \mathcal{O} \xrightarrow{s_B} B \to L^* \otimes \mathcal{I}_2 \to 0, \tag{3.2}$$

with $\mathcal{I}_1$, $\mathcal{I}_2$ the ideal sheaves of 1 and 2 points in $C \subset S$, respectively. We take the one point to be one of the two points, as in the last example.

Setting

$$A' := \pi^* A \otimes \mathcal{O}(3S) \otimes \gamma, \quad B' := \pi^* B \otimes \mathcal{O}(-3S) \otimes \gamma, \quad E := A' \oplus B',$$

we can, as in the last two examples, choose compatible Kähler forms on S and X such that A and B are slope-stable and A', B' have degree zero, so that E is slope-polystable. Of course $\Lambda^4 E$ is trivial and $\chi(E) = 3$.

Then

$$H^1_X(\Lambda^2 A') = H^1_X((\Lambda^2 B')^*) = H^1_S(L \otimes \pi_* \mathcal{O}(6S)) = H^1_S(L)^{\oplus 6} = 0,$$

and

$$H^1_X(\Lambda^2 B') = H^1_X((\Lambda^2 A')^*) = H^0_S(L^* \otimes R^1 \pi_* \mathcal{O}(-6S)) = H^0_S(L^* \otimes \gamma)^{\oplus 6} = 0,$$

so that as before the pairing on $\Lambda^2 E$ reduces to its $H^1(A' \otimes B')$ summand, and similarly for the dual pairing and the twists by flat line bundles.

The usual arguments give the vanishing of $H^0(A \otimes B)$, its dual and their twists by γ, so that the pairing reduces to 18 copies of

$$H^1_S(\pi_* \mathcal{O}_X \otimes A \otimes B) \otimes H^1_S(A \otimes \gamma) \otimes H^0_S(B) \to \mathbb{C}\,,$$

and the dual to 18 copies of

$$H^1_S(\pi_* \mathcal{O}_X \otimes A^* \otimes B^*) \otimes H^0_S(A^*) \otimes H^1_S(B^* \otimes \gamma) \to \mathbb{C}\,.$$

On twisting E by any flat line bundle with a non-zero α or β component the $\pi_* \mathcal{O}_X$ term vanishes so the pairing does too. Thus we need only consider twists by γ.

Thus by Serre duality the pairing is equivalent to

$$H^1(A \otimes \gamma)^{\oplus 2} \xrightarrow{s_B \oplus s'_B} H^1(A \otimes B \otimes \gamma),$$

where we have noted from (3.2) that $H^0(B) \cong \mathbb{C}^2$ and we have picked a basis s_B, s'_B. Similarly the dual pairing is represented by

$$H^1(B^* \otimes \gamma) \xrightarrow{s_{A^*}} H^1(A^* \otimes B^* \otimes \gamma).$$

We want these to be non-zero, but zero on twisting by γ.

For the first pairing we tensor (3.2) by $A(\otimes\gamma)$, giving

$$0 \to H^1(A(\otimes\gamma)) \xrightarrow{s} H^1(A \otimes B(\otimes\gamma)),$$

where s is any section of B (which we see from (3.2) vanishes on 2 points in $C \subset S$). Thus it is sufficient to show that $H^1(A \otimes \gamma) \neq 0$ and $H^1(A) = 0$. But $h^1(A \otimes \gamma) = -\chi(A \otimes \gamma) + h^0(A \otimes \gamma) + h^0(A^*) = 0 + 0 + 1 = 1$, and $h^1(A) = -\chi(A) + h^0(A) + h^0(A^* \otimes \gamma) = 0 + 0 + 0 = 0$.

For the dual pairing, tensoring (3.1) with $B^*(\otimes\gamma)$ yields

$$0 \to H^1(B \otimes \mathcal{I}_1(\otimes\gamma)) \to H^1(B^*(\otimes\gamma)) \xrightarrow{s_{A^*}} H^1(A^* \otimes B^*(\otimes\gamma)).$$

With the γ the last map is the dual pairing, and by Riemann-Roch and $h^0(B) = 2$, $\chi(B) = -1$ we see the sequence is

$$0 \to 0 \to \mathbb{C}^3 \xrightarrow{s_{A^*}} H^1(A^* \otimes B^* \otimes \gamma)$$

so that the pairing is indeed non-zero. Removing the γs gives the twisted dual pairing, which we would like to show vanishes.

Since the zero of s_{A^*} was chosen to lie in the two zeros of s_B, we see that $h^0(B \otimes \mathcal{I}_1) \geq 1$, while by Riemann-Roch $h^1(B^*) = 1 + h^0(B^*) + h^0(B \otimes \gamma) = 1$, so that the sequence becomes

$$0 \to \mathbb{C} \to \mathbb{C} \xrightarrow{s_{A^*}} H^1(A^* \otimes B^*)$$

and the final map must be zero.

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